On the number of orthogonal systems in vector spaces over finite fields

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Abstract

Iosevich and Senger (2008) showed that if a subset of the d-dimensional vector space over a finite field is large enough, then it contains many k-tuples of mutually orthogonal vectors. In this note, we provide a graph theoretic proof of this result.

1 Introduction

A classical set of problems in combinatorial geometry deals with the question of whether a sufficiently large subset of $\mathbb{R}^d$, $\mathbb{Z}^d$ or $\mathbb{F}_q^d$ contains a given geometric configuration. In a recent paper [3], Iosevich and Senger showed that a sufficiently large subset of $\mathbb{F}_q^d$, the d-dimensional vector space over the finite field with q elements, contains many k-tuple of mutually orthogonal vectors. Using geometric and character sum machinery, they proved the following result (see [3] for the motivation of this result).

Theorem 1.1 *([3]) Let $E \subset \mathbb{F}_q^d$, such that*

$$|E| \geqslant Cq^{d\frac{k-1}{k}+\frac{k-1}{2}+\frac{1}{k}} \tag{1.1}$$

with a sufficiently large constant $C > 0$, where $0 < \binom{k}{2} < d$. Let λ_k be the number of k-tuples of k mutually orthogonal vectors in E. Then

$$\lambda_k = (1+o(1))\frac{|E|^k}{k!}q^{-\binom{k}{2}}. \tag{1.2}$$

In this note, we provide a different proof to this result using graph theoretic methods. The main result of this note is the following.

Theorem 1.2 *Let $E \subset \mathbb{F}_q^d$, such that*

$$|E| \gg q^{\frac{d}{2}+k-1}, \tag{1.3}$$

where $d \geqslant 2k-1$. Then the number of k-tuples of k mutually orthogonal vectors in E is

$$(1+o(1))\frac{|E|^k}{k!}q^{-\binom{k}{2}}. \tag{1.4}$$

Note that Theorem 1.1 only works in the range $d \geqslant \binom{k}{2}$ (as larger tuples of mutually orthogonal vectors are out of range of the methods uses) while Theorem 1.2 works in a wider range $d \geqslant 2k-1$. Moreover, Theorem 1.2 is stronger than Theorem 1.1 in the same range.

2 Proof of Theorem 1.2

We call a graph $G=(V,E)$ (n,d,λ)-graph if G is a d-regular graph on n vertices with the absolute values of each of its eigenvalues but the largest one is at most λ. It is well-known that if $\lambda \ll d$ then an (n,d,λ)-graph behaves similarly as a random graph $G_{n,d/n}$. Let H be a fixed graph of order s with r edges and with automorphism group $\mathrm{Aut}(H)$. Using the second moment method, it is not difficult to show that for every constant p the random graph $G(n,p)$ contains

$$(1+o(1))p^r(1-p)^{\binom{s}{2}-r}\frac{n^s}{|\mathrm{Aut}(H)|} \tag{2.1}$$

induced copies of H. Alon extended this result to (n,d,λ)-graphs. He proved that every large subset of the set of vertices of an (n,d,λ)-graph contains the "correct" number of copies of any fixed small subgraph (Theorem 4.10 in [2]).

Theorem 2.1 *([2]) Let H be a fixed graph with r edges, s vertices and maximum degree Δ, and let $G=(V,E)$ be an (n,d,λ)-graph, where, say, $d \leqslant 0.9n$. Let $m<n$ satisfies $m \gg \lambda\left(\frac{n}{d}\right)^{\Delta}$. Then, for every subset $U \subset V$ of cardinality m, the number of (not necessarily induced) copies of H in U is*

$$(1+o(1))\frac{m^s}{|\mathrm{Aut}(H)|}\left(\frac{d}{n}\right)^r. \tag{2.2}$$

Note that the above theorem is stated for simple graphs in [2] but there is no different in the proof if we allow loops in the graph G.

We recall a well-known construction of Alon and Krivelevich [1]. Let $PG(q,d)$ denote the projective geometry of dimension $d-1$ over finite field $\mathbb{F}_q$. The vertices of $PG(q,d)$ correspond to the equivalence classes of the set of all non-zero vectors $x=(x_1,\ldots,x_d)$ over $\mathbb{F}_q$, where two vectors are equivalent if one is a multiple of the other by an element of the field. Let $G_P(q,d)$ denote the graph whose vertices are the points of $PG(q,d)$ and two (not necessarily distinct) vertices x and y are adjacent if and only if $x_1y_1+\ldots+x_dy_d=0$.

This construction is well known. In the case $d = 2$, this graph is called the Erdös-Rényi graph. It is easy to see that the number of vertices of $G_P(q, d)$ is $n_{q,d} = (q^d - 1)/(q - 1)$ and that it is $d_{q,d}$-regular for $d_{q,d} = (q^{d-1} - 1)/(q - 1)$. The eigenvalues of G are easy to compute ([1]). Let A be the adjacency matrix of G. Then, by properties of $PG(q, d)$, $A^2 = AA^T = \mu J + (d_{q,d} - \mu)I$, where $\mu = (q^{d-2} - 1)/(q - 1)$, J is the all one matrix and I is the identity matrix, both of size $n_{q,d} \times n_{q,d}$. Thus the largest eigenvalue of A is $d_{q,d}$ and the absolute value of all other eigenvalues is $\sqrt{d_{q,d} - \mu} = q^{(d-2)/2}$.

Now we are ready to give a proof of Theorem 1.2. Let $G(q, d)$ denote the graph whose vertices are the points of $\mathbb{F}_q^d - (0, \ldots, 0)$ and two (not necessarily distinct) vertices x and y are adjacent if and only if they are orthogonal, i.e. $x_1y_1 + \ldots + x_dy_d = 0$. Then $G(q, d)$ is just the product of $q - 1$ copies of $G_P(q, d)$. Therefore, it is easy to see that the number of vertices of G is $N_{q,d} = (q - 1)n_{q,d} = q^d - 1$ and that it is $D_{q,d}$-regular for $D_{q,d} = (q - 1)d_{q,d} = q^{d-1} - 1$. The eigenvalues of $G(q, d)$ are also easy to compute. Let V be the adjacency matrix of $G(q, d)$. Then by the properties of $PG(q, d)$,

$$V^2 = VV^T = \rho J_{N_{q,d}} + (D_{q,d} - \rho) \bigoplus_{n_{q,d}} J_{q-1}, \tag{2.3}$$

where $\rho = (q-1)\mu = q^{d-2} - 1$, $J_{N_{q,d}}$ is the all one matrix of size $N_{q,d} \times N_{q,d}$ and J_{q-1} is the all one matrix of size $(q - 1) \times (q - 1)$. Thus, all eigenvalues of V^2 are all eigenvalues of $(q-1)\rho J_{n_{q,d}} + (q-1)(D_{q,d} - \rho)I_{n_{q,d}}$ and zeros (with $J_{n_{q,d}}$ is the all one matrix and $I_{n_{q,d}}$ is the identity matrix, both of size $n_{q,d} \times n_{q,d}$). Therefore, the largest eigenvalue of V is $D_{q,d}$ and the absolute values of all other eigenvalues are either $\sqrt{(q - 1)(D_{q,d} - \rho)} = (q - 1)q^{(d-2)/2}$ or 0. This implies that $G(q, d)$ is a $(q^d - 1, q^{d-1} - 1, (q - 1)q^{(d-2)/2})$-graph. Theorem 1.2 now follows immediately from Theorem 2.1.

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